Signaling with Costly Acquisition of Signals

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Abstract

In this paper we identify a novel reason why signaling may fail to separate types, which is specific to cases where the receiver has to incur a cost to acquire the signal sent by the sender. If the receiver chooses not to incur the acquisition cost, then all sender's types find it optimal to pool on the least costly signal; also, if all sender's types pool on the least costly signal, then the receiver finds it optimal not to incur the acquisition cost. This kind of strategic complementarity makes the resulting pooling equilibrium extremely robust, even when costs of signal acquisition are very small. Also, pooling is shown to be robust to all refinements based on out-of-equilibrium beliefs, even when the sender can engage in further signaling that can act as an "invitation" to acquire the main signal, and when acquisition costs are smooth and depend on the receiver's effort to acquire the signal. However, the pooling outcome is not necessarily robust when signals are not purely costly or the receiver can initially commit to acquire the signal. These results provide a new source of interest in pooling equilibria.

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1   Introduction

Signaling is a pervasive phenomenon in economic interactions, emerging in many situations where there are information asymmetries. Many signaling models have been developed and studied, making the class of signaling games a quite prominent one in economics (see Riley, 2001, for a comprehensive survey).

An important characteristic of signaling games is that they typically show many Bayes-Nash equilibria with rather distinct features: equilibria in which sender's types pool together by sending the same signal and equilibria where sender's types separate from each other by sending different signals.

In applied research, signaling models are often used with the focus on the best separating equilibrium, also called the Riley equilibrium, i.e., the equilibrium where all sender's types separate but they incur the minimun necessary signaling cost to do so. This is in good part due to an important stream of literature that has shown the prominence of the Riley equilibrium when agents are supposed to possess a sufficient degree of forward induction (see Sobel, 2009, for an instructive survey and Subsection 6.1 where we provide more details on this point).

Quite surprisingly, however, not much attention has been given to the possibility that the acquisition of the signal by the receiver might be a costly activity. Is the assumption of freely acquisition of signals innocuous? In this paper we show that it is definitely not so. Indeed, even a very small cost of signal acquisition can make a great difference in terms of the robustness (and plausibility) of equilibria. In particular, we show that in the presence of costs to acquire the signal the pooling of sender's types becomes at least as prominent as their separation as an equilibrium outcome. This happens because of the emergence of strategic complementarity: if the receiver chooses not to incur the acquisition cost, then all sender's types find it optimal to pool on the minimum signal and, at the same time, if the different types of the sender pool on the same signal, then the receiver finds it optimal not to incur the acquisition cost. So, a complementarity naturally arises between the receiver's incentive to *costly* acquire the signal and the sender's incentive to engage in the *costly* signaling activity. Our results suggest that new attention should be given to pooling outcomes. This could have far-reaching implications, especially in the light of the widespread reliance on separating equilibria in applied models.

The paper is organized as follows. In Section 2 we review the literature on costly acquisition of information. In Section 3 we introduce signaling games with costly acquisition of signals by means of an example that is a variant of the classical Spence's signaling model. In Section 4 we define a general class of signaling games with costly acquisition of signals. In Section 5 we show that the existence of acquisition costs can lead to kind of strategic complementarity that sustains a pooling equilibrium, which is also shown to be the unique pooling outcome; further, we contrast

the equilibra that emerge in this class of games with those that emerge in the related class of standard signaling games (i.e., with no acquisition costs). In Section 6 we explore the robustness of the pooling outcome along four dimensions: equilibrium refinements acting on out-of-equilibrium beliefs (Subsection 6.1), equilibrium selection by means of further signaling that can act as an "invitation" to acquire the main signal (Subsection 6.2), smooth acquisition costs that depend on the receiver's acquisition effort (Subsection 6.3), signals that are not necessarily a net cost for the sender (Subsection 6.4), and the possibility for the receiver to initially commit to acquire the signal (6.5). We find that only in the last two cases the pooling outcome might not survive, restoring the prominence of the separating outcome, while it is robust in the first three cases. We also point out that, for the case in which signals are not purely costly, our findings are still relevant if we want to predict the effects of imposing a mandatory minimum signal to the senders.

2 Related literature

The idea that the acquisition of information is a strategic choice which comes at a cost is receiving increasing attention in economics. Several models with this feature have been investigated but just a few of them are closely related to our model. In fact, most of these models do not consider a sender-receiver setup, and those who consider a signaling framework do not explore the possibility that information acquisition involves the signal itself.[1]

Gabaix et al. (2006) test the directed cognition model – which assumes that agents use partially myopic option-value calculations to select their next cognitive operation – by studying information acquisition in two different experiments. Caplin and Dean (2014) develop a revealed preference test for the costly acquisition of information, encompassing models of rational inattention, sequential signal processing, and search. Liu (2011) studies the dynamic behavior of firms and customers in markets with costly acquisition of information on past transactions. Brocas et al. (2012) consider a model where two adversaries can spend resources in acquiring public information about the unknown state of the world in order to influence the choice of a decision maker and find that as the cost of information acquisition for one adversary increases, he collects less evidence whereas the other adversary collects more evidence. Shi (2012) analyzes optimal auction design in a setting where values are private and there are several potential buyers who can each costly acquire information about others' valuations prior to participation in the auction. Oliveros (2013) studies the role of abstention in an election where voters can costly acquire information and the cost increases in the precision of the information acquired. Morath and Münster (2013) study the incentives

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[1]See for instance the literature on strategic pricing and information acquisition of product quality from a third party started with Bester and Ritzberger (2001) (see also Gertz, 2014 and Martin, 2015).

for information acquisition ahead of conflicts, showing that information acquisition is excessive or not depending on public observability. Colombo et al. (2014) consider a general framework where agents interact under both strategic complementarities/substitutabilities and externalities, showing that the social value of public information is affected crucially by the private choice to costly acquire information. Argenziano et al. (2014) study the case of an expert that can costly acquire information to advise a decision-maker; they show that the information acquired by the expert can be more precise than the information that would be directly acquired by the decision maker, even if the expert is not more efficient than the decision maker at acquiring information. Menichini and Simmons (2014) characterize the optimal audit structure with ex-ante information acquisition.

A paper more closely related to ours is Dewatripont and Tirole (2005) which develops a theory of costly communication where both the sender and the receiver have to incur a cost in order to communicate.[2] The model can be seen as a standard cheap talk model where the precision of the message depends on the cost sustained by the sender and where the receiver has to incur a cost – which may depend on the message precision – to acquire the message. Due to such costs, a form of strategic complementarity arises – that is similar to the one emerging in our model – which gives rise to a robust babbling equilibrium where the message sent by the sender contains no information and the receiver does not acquire it. The main difference with our model is that Dewatripont and Tirole (2005) do not consider costly signaling, but different modes of cheap communication.[3]

Tirole (2009) develops a model of limited cognition and examines its consequences for contractual design. This paper formalizes the idea that the acquisition of information is a costly activity because of cognitive limitation. This same idea is applied to persuasion in Bilancini and Boncinelli (2014a) where the receiver has to incur a cognitive cost to fully and precisely elaborate information. In this model the sender tries to persuade the receiver to accept an offer by sending a costly signal - the reference cue - which refers the offer to a category of offers whose average quality is known by the receiver; the actual quality is a sender's private information, but the receiver can pay a cost to acquire it – the elaboration cost. Pooling equilibria emerge also in this setup and turn out to be robust. However, they are not due to strategic complementarity: a pooling equilibrium – i.e., an equilibrium where the bad offer and the good offer are proposed with the same reference cue – is sustained by the fact that the receiver accepts or not the offer independently of the observed reference cue, no matter what information she decides to acquire. An important difference between

the model by Bilancini and Boncinelli (2014a) and the one developed in the present paper is that in the former the acquisition cost is paid to acquire hard information on the state of the world, while in the present paper the cost is paid to acquire the soft information embodied by the signal.

3   A motivating example

Consider the following simple variant of the classical model by Spence (1973). There is one employer E that wants to hire a worker W. There are two types of workers, distinguished by their productivity $\theta \in \{1, 2\}$, which is a worker's private information; E has a prior $0 < p < 1$ that W is highly productive, i.e., that $\theta = 2$. Technology and market conditions are such that E's net profits are given by $\theta - w$ if a worker is hired, with w the wage paid to the hired worker and θ his productivity, while otherwise profits are 0.

Moreover, W can acquire education by incurring a cost that is type-dependent. In particular, suppose that W comes from a foreign country and that he has to move to E's country in order to be hired. Suppose also that W can only acquire education in the foreign country, and that the only available alternatives are a good school G and a bad school B, which are not previously known to E. For the prospective worker of type θ, the cost of attending G is $2/\theta$ and the cost of attending school B is $1/\theta$. So, attending school G is more costly than attending school B, and it is relatively more so for the low type $\theta = 1$. This provides W with a costly signal $x \in \{G, B\}$ that potentially allows W's types to separate.

So far, there is no substantial difference from Spence's model. However, what if E, in order to assess the quality of the schooling signal x sent by W, has to actively acquire the information on what school W has attended in the country he comes from, and what attendance costs have been paid? These information can well not come for free and, we stress, this can make the difference. We observe that the costs of acquiring such information can be interpreted as due to the material and/or the cognitive effort which is necessary to retrieve and elaborate the relevant data on x. On the material side, E might have to search and collect information on G and B, and maybe also pay to translate documents that would be otherwise unaccessible. On the cognitive side, E might have to make an effort to elaborate the collected information in order to establish that one school is G with costs 2θ and the other is B with costs θ and to assign the signal x to either G or B – otherwise the schools would be undistinguishable to E. To model this situation suppose that E has to pay a cost $c > 0$ to acquire the signal x sent by W. In particular, if E does not incur the cost c, then E cannot condition his actions on x.

Consider now the following situation: W chooses B independently of her type, i.e., $x(1) = x(2) = B$, and E decides not to acquire the signal x. It is easy to check that this is an equilibrium

in the present example: both types of W strictly lose by switching to the more costly G, and E strictly loses by acquiring the signal because it costs c and provides no new information. We observe that such an equilibrium is very similar to the pooling equilibrium with lowest signal that emerges in Spence's model. However, we stress that the presence of acquisition costs makes this pooling equilibrium much more robust than that pooling equilibrium in Spence's model. Since E does not acquire any signal, W cannot use out-of-equilibrium signals to communicate with E, and the reason is that W would not even notice that such signals have been sent. In particular, even if W's high type deviates from $x(2) = B$ to $x(2) = G$, there is no way to let E know – or even imagine – about such a deviation. So, arguments based on the reasonability of out-of-equilibrium beliefs cannot refine away this pooling equilibrium.

There is another important difference. In this example the lowest signal pooling equilibrium is the *only* pooling equilibrium, again in contrast with Spence's model where there are multiple pooling equilibria. To see why, consider the case where both types of W pool on G, i.e., $x(1) = x(2) = G$. Given this behavior by W, E finds it strictly profitable *not to incur* the acquisition cost, as acquiring the signal provides no new information. But if E does not acquire the signal x, then the choice of $x(1) = x(2) = G$ cannot be sustained in equilibrium since each of W's type would strictly gain by switching from G to B, as this allows to save on the cost of signaling without adversely affecting E's beliefs.

4 The model

We now introduce the more general game of *signaling with costly acquisition of signals* (SCAS). There is one sender S and one receiver R (sometimes referred to as "he" and "she", respectively). The sender S observes his own type $t \in T$, with T a finite set of cardinality n, and then chooses a signal $x \in X = \mathbb{R}_+$.[4] The receiver R can exert costly effort and acquire the signal x, or save on effort and observe nothing. We denote with $s \in \{s1, s2\}$ such a choice, where $s1$ means that x is not acquired and $s2$ means that x is acquired and effort is exerted.[5] In any case, then R has to take an action $y \in Y = \mathbb{R}$. The prior beliefs held by R on T are given by $p = (p_1, ..., p_n) \in \Delta T$ where p_t denotes the probability that S is of type $t \in T$.

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[4]We have chosen to use continuous action spaces because this is the most common setting for signaling models. We stress that our results might be stated, with few straightforward adjustments, using discrete action spaces too.

[5]This labeling owes to the classification of elaboration processes as "System 1", or S1, which is fast, cheap and intuitive, and "System 2", or S2, which is slow, costly and analytical (see, e.g., Kahneman, 2003). We stress this interpretation based on cognitive effort because we think that it applies to many relevant cases of signal acquisition. Of course, other interpretations are possible where the cost of acquiring the signal is entirely due to non-psychological factors.

Utility for S is $U : T \times X \times Y \to \mathbb{R}$, and utility for R is $V : T \times X \times Y \times \{s1, s2\} \to \mathbb{R}$. The following assumptions on utility functions hold:

A1. *continuity:* U and V are continuous over x and y;

A2. *monotonicity in action:* U is strictly increasing in y;

A3. *costly signaling:* U is strictly decreasing in x;

A4. *single-crossing property:* $U(t, x, y) \leq U(t, x', y')$, with $x' > x$, implies that $U(t', x, y) < U(t', x', y')$ for all $t' > t$ and $y, y' \in Y$;

A5. *fixed positive cost of acquiring the signal:*
$V(t, x, y, s1) - V(t, x, y, s2) = c > 0$ for all $t \in T$, $x \in X$, $y \in Y$.

In the light of A5, we can define function $v : T \times X \times Y \to \mathbb{R}$ such that $v(t, x, y) - c = V(t, x, y, s2)$, which represents R's utility gross of the acquisition cost.

A strategy for S is a function $\mu : T \to X$;[6] we denote with $\mathcal{M}$ the set of all possible μ. A strategy for R is a pair (s, α) where $s \in \{s1, s2\}$ and $\alpha : X \times \{s1, s2\} \to Y$ is a function such that $\alpha(x, s1) = \alpha(x', s1)$ for all $x, x' \in X$, i.e., R's action is unconditional on x whenever $s = s1$ is chosen; we denote with $\mathcal{A}$ the set of all such functions.[7]

For given μ and (s, α), R has posterior beliefs that crucially depend on her choice of s. If R chooses $s = s2$ then she has posterior beliefs $\beta(x|\mu, s2) = (\beta_1(x|\mu, s2), \ldots, \beta_n(x|\mu, s2)) \in \Delta T$, where each $\beta_t(x|\mu, s2)$ denotes the probability that S is of type t conditional on the observation of x. These beliefs can be obtained by Bayes's rule, if applicable, or be chosen otherwise. If, instead, R chooses $s = s1$ then she can only rely on her priors – no new information is acquired – so that posteriors are trivially identical to priors: $\beta_t(x|\mu, s1) = \beta_t(x'|\mu, s1) = p_t$, for all $t \in T$ and all $x, x' \in X$.

We introduce the following additional assumption:

A6. *uniqueness of best action under s1:*
$\rho^{s1}(\mu) = \arg\max_{y \in Y} \sum_{t \in T} p_t v(t, \mu(t), y)$ exists and is single valued.

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[6]This notation is somewhat non-standard, but it allows us to be consistent with the related literature (e.g., Carlsson and Dasgupta, 1997).

[7]Here we adopt a specification of strategies for R that more is compact but perhaps less straightforward than, e.g., $\alpha(s, \alpha^{s_1}, \alpha^{s_2})$ where $\alpha^{s_1} \in Y$ is the action taken if $s = s_1$ and $\alpha^{s_1} : X \to Y$ is the action taken if $s = s_2$. We stress that the two formulations are equivalent in the sense that we can go from one to the other by coalescing (Thompson, 1952) the information set that R reaches when $s = s_1$ with the one where R chooses s, although in this game there is an infinite number of information sets. For more details see Elmes and Reny (1994) and the discussion in Alós-Ferrer and Ritzberger (2013, example 9).

Assumption A6 resembles an assumption that is typically made in standard signaling models: the single-valuedness of the receiver's best reply. We stress, however, that A6 does not ensure this much. In fact, it is consistent with the case where R is indifferent between choosing $\rho^{s1}(\mu)$ with no acquisition of the signal and some other action (or actions) with the acquisition of the signal. We observe that, because of the separability of acquisition costs implied by assumption A6, the best action does not depend directly on the choice between $s1$ and $s2$, but it does indirectly through the updating of beliefs that becomes possibile when $s2$ is chosen. This implies that the best action against μ is given by $\rho^{s1}(\mu)$ whenever posteriors are identical to priors, independently of the choice of $s \in \{s1, s2\}$.

DEFINITION **1.** *(weak Perfect Bayes-Nash equilibrium of the SCAS game)*
A wPBE equilibrium of a SCAS game is a profile of strategies $(\mu, (s, \alpha))$ such that:

E1. $\mu(t) \in \underset{x \in X}{\arg\max}\, U(t, x, \alpha(x, s))$, *for all $t \in T$;*

E2. *for all $x \in X$, there exists beliefs $\beta(x|\mu, s) \in \Delta T$ such that (s, α) satisfies:*

    E2.1. $\alpha(x, s1) = \rho^{s1}(\mu)$ *for all $x \in X$;*

    E2.2. $\alpha(x, s2) \in \underset{y \in Y}{\arg\max} \sum_{t \in T} \beta_t(x|\mu, s2) v(t, x, y) - c$ *for all $x \in X$;*

    E2.3. $s = s1$ *implies that:*
$$\sum_{t \in T} p_t v(t, \mu(t), \rho^{s1}(\mu)) \geq \sum_{t \in T} p_t \left(\sum_{k \in T} \beta_k(\mu(t)|\mu, s2) v(t, \mu(t), \alpha(\mu(t), s2)) - c \right);$$

    E2.4. $s = s2$ *implies that:*
$$\sum_{t \in T} p_t v(t, \mu(t), \rho^{s1}(\mu)) \leq \sum_{t \in T} p_t \left(\sum_{k \in T} \beta_k(\mu(t)|\mu, s2) v(t, \mu(t), \alpha(\mu(t), s2)) - c \right);$$

E3. *the beliefs $\beta(x|\mu, s) \in \Delta T$ are calculated by means of Bayes's rule whenever possible.*

The meaning of E1 is straightforward: S must be best-replying to R. Similarly, the meaning of E2 is that R must be best-replying to S given her beliefs; in particular, R must be optimally choosing action $y \in Y$ under both $s1$ (E2.1) and $s2$ (E2.2) (and, in the latter case, for any observed signal x) as well as choosing optimally between $s1$ and $s2$ (E2.3 and E2.4). Condition E3 is also straightforward. We observe that, in the present setup, posterior beliefs along the equilibrium path are the following:

- if $s = s2$ then, for all $t \in T$ and for all x such that $\mu(t') = x$ for some $t' \in T$:

$$\beta_t(x|\mu, s) = \begin{cases} \dfrac{p_t}{\sum_{k:\mu(k)=x} p_k} & \text{if } \mu(t) = x \\[2mm] 0 & \text{if } \mu(t) \neq x; \end{cases}$$

- if $s = s1$ then $\beta_t(x|\mu, s) = p_t$ for all $t \in T$ and for all $x \in X$.

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Lastly, in order to better contrast our results with the results on standard signaling games we find it useful to define the standard signaling game that can be obtained from a SCAS game by forcing R to play $s = s2$ and setting $c = 0$. We call this a game of *signaling with free acquisition of signals* (SFAS). Note that a SFAS game with utilities U and v is actually the standard signaling game – i.e., with no costs to acquire the signal – that can be obtained from a SCAS game with utilities U and v and any acquisition cost $c > 0$. In the light of this, we denote with $\Gamma(T, p, U, v, c)$ a given SCAS game – where T is the sender's type space and p is the tuple of receiver's priors – and with $\Gamma(T, p, U, v, 0)$, its associated SFAS game.

5 Equilibria

The set of Bayes-Nash equilibria of a SCAS game is in general different from the set of Bayes-Nash equilibria of a typical signaling game. This difference is mostly due to the inexistence of pooling equilibria where all sender's types pool on non-minimum signals. To make this claim precise we provide a number of results characterizing the set of equilibria of a generic SCAS game, and we compare it to the set of equilibria of the associated SFAS game.

5.1 Strategic complementarity leads to pool on the null signal

Our first result states that in a SCAS game a pooling equilibrium – i.e., an equilibrium where all sender's types send the same signal – must be such that sender's types pool on the signal $x = 0$ and the receiver does not acquire the signal, implying that there is a *unique pooling outcome in equilibrium* – to which we sometimes refer as a *no-signal pooling equilibrium*. The following proposition formalizes:

PROPOSITION 1. *A SCAS game* $\Gamma(T, p, U, v, c)$ *has a pooling equilibrium. If* $(\mu^P, (s^P, \alpha^P))$ *is a pooling equilibrium, then it must be such that* $\mu^P(t) = 0$ *for all* $t \in T$, $s^P = s1$, *and* $\alpha^P(x, s1) = \alpha^P(0, s2) = \rho^{s1}(\mu^P)$ *for all* $x \in X$.

Proof. We first show that the profile $(\mu^P, (s^P, \alpha^P))$ is an equilibrium. Preliminarily, note that by A6 (uniqueness of best action under $s1$) R's expected utility $\sum_{t \in T} p_t v(t, \mu^P(t), y)$ admits a maximum over Y and, hence, the profile $(\mu^P, (s^P, \alpha^P))$ exists. For notational convenience we denote this maximum with $y^* = \rho^{s1}(\mu^P)$.

Consider R deviating from (s^P, α^P). Since $\alpha^P(x, s1) = y^*$ for all $x \in X$, no strictly profitable deviation from α^P exists as long as R maintains $s1$. Consider a deviation to (s', α') with $s' = s2$ and some $\alpha' \in \mathcal{A}$. We observe that, since $\mu^P(t) = 0$ for all $t \in T$, R obtains no additional information by playing $s2$ instead of $s1$, and therefore her posterior beliefs must be equal to her priors p. This

implies that y^* is an optimal action also when $s' = s2$ and signal $\mu^P(t) = 0$, for all $t \in T$, is observed. Since by A5 (fixed positive acquisition cost) the only effect on utility of playing $s2$ instead of $s1$ is, for a given choice of $\alpha \in \mathcal{A}$, to incur the constant cost $c > 0$, it follows that R's expected utility is lower under deviation $(s2, \alpha')$, for all $\alpha' \in \mathcal{A}$, than under (s^P, α^P).

Consider S deviating from μ^P. In particular, consider S deviating to μ' such that $\mu'(t') > 0$ for some $t' \in T$. Recall that $\alpha^P(x, s1) = y^*$ for all $x \in X$, i.e., the action chosen by R is y^* independently of the actual value of $\mu'(t)$, $t \in T$. This, together with assumption A3 (costly signaling) implies that S's expected utility cannot be greater under any $\mu' \in \mathcal{M}$ than under μ^P.

We now show that no pooling equilibrium other than $(\mu^P, (s^P, \alpha^P))$ exists. Consider the profile $(\mu^{P'}, (s^{P'}, \alpha^{P'}))$ where $\mu^{P'}(t) = x^{P'} > 0$ for all $t \in T$. Note that, exactly because $\mu^{P'}(t) = x^{P'}$ for all $t \in T$, along the equilibrium path R never learns anything and so R takes the same action $y^{P'} = \alpha^{P'}(\mu^{P'}(t), s^{P'})$ for all $t \in T$ (note that $y^{P'}$ may or may not be equal to y^*). If $s^{P'} = s2$, then – by assumption A5 – R's gets an expected utility of $\sum_{t \in T} p_t v(t, x^{P'}, y^{P'}) - c$ which is strictly lower than $\sum_{t \in T} p_t v(t, x^{P'}, y^{P'})$, i.e., the expected utility that R obtains by playing $s1$ together with any $\alpha^{P''} \in \mathcal{A}$ such that $\alpha^{P''}(x, s1) = y^{P'}$ for all $x \in X$. So, in order for $(s^{P'}, \alpha^{P'})$ to be a best reply for R to $\mu^{P'}$, it must be that $s^{P'} = s1$ and, hence, $\alpha^{P'}(x, s1)$ must be constant over X and, in particular, such that $\alpha^{P'}(x, s1) = \rho^{s1}(\mu^{P'})$ for all $x \in X$. But if this is the case, then we claim that S has a profitable deviation. In particular, consider S deviating to μ^P. Since R always responds with $\rho^{s1}(\mu^{P'})$ and by A3 (costly signaling), it follows that S's expected utility is strictly greater under μ^P than under $\mu^{P'}$. $\qquad\square$

5.2 Sufficiently low acquisition costs allow separation

Proposition 1 establishes that the presence of a positive cost to acquire the signal – no matter how small – induces a strong reduction in the number and variety of pooling equilibria, actually leading to a unique outcome where all sender's types pool on the null signal. Our second result shows that such a strong reduction does not take place for separating and semi-separating equilibria. More precisely, if a SFAS game has a separating or semi-separating equilibrium and the information on sender's types is of some value to the receiver, then also all associated SCAS games with acquisition costs sufficiently low possess the same equilibrium. We also observe that, trivially, all separating or semi-separating equilibria of a SCAS game must be, a fortiori, equilibria of the associated SFAS game.[8]

In a SFAS game $\Gamma(T, p, U, v, 0)$, given $\mu \in \mathcal{M}$ where $\mu(t) \neq \mu(t')$ for some $t, t' \in T$, we say that

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[8]Driving the acquisition cost to 0 affects all sender's strategies encompassing the acquisition of the signal in the same way, leaving unaltered their relative desirability for S and making them relatively more desirable with respect to strategies not encompassing the acquisition of the signal.

the information conveyed by μ on sender's types is valuable to R if there exits $\bar{v} > 0$ such that we have:

$$\max_{\alpha \in \mathcal{A}} \sum_{t \in T} \beta_t(\mu(t)|\mu, s2)v(t, \mu(t), \alpha(\mu(t)), s2) - \max_{y \in Y} \sum_{t \in T} p_t v(t, \mu(t), y) \geq \bar{v}; \tag{1}$$

i.e., R gains at least $\bar{v}$ by acquiring information on types conveyed by μ. We observe that if the information conveyed by μ is valuable to R in the SFAS game $\Gamma(T, p, U, v, 0)$, then it is valuable also in the SCAS game $\Gamma(T, p, U, v, c)$, for all $c > 0$.

The next proposition formalizes the result mentioned above:

PROPOSITION **2.** *Let $(\mu^S, (s^S, \alpha^S))$ be an equilibrium profile of the SFAS game $\Gamma(T, p, U, v, 0)$ where $\mu^S(t) \neq \mu^S(t')$ for some $t, t' \in T$. If the information converyed by μ^S on sender's types is valuable to R, then there exists $\bar{c}\left((\mu^S, (s^S, \alpha^S))\right) > 0$ such that $(\mu^S, (s^S, \alpha^S))$ is an equilibrium of all SCAS games $\Gamma(T, p, U, v, c)$ with $c \leq \bar{c}\left((\mu^S, (s^S, \alpha^S))\right)$.*

Proof. Let $(\mu^S, (s^S, \alpha^S))$ be an equilibrium of $\Gamma(T, p, U, v, 0)$ with supporting beliefs $\beta^S(x|\mu, s) \in \Delta T$. Since $\Gamma(T, p, U, v, 0)$ is a SFAS game, it must be that $s^S = s2$. Moreover, since $(\mu^S, (s^S, \alpha^S))$ is an equilibrium of $\Gamma(T, p, U, v, 0)$, μ^S must be a best-reply to (s^S, α^S) for S.

Consider now the SCAS game $\Gamma(T, p, U, v, c)$. Note that $(\mu^S, (s^S, \alpha^S))$ is a profile which is feasible also in $\Gamma(T, p, U, v, c)$, for any $c > 0$. Note also that, by construction, S has the same set of strategies and faces the same payoffs in game $\Gamma(T, p, U, v, 0)$ and game $\Gamma(T, p, U, v, c)$, for any $c > 0$. Hence, if μ^S is a best reply to (s^S, α^S) for S in $\Gamma(T, p, U, v, 0)$ then it is also a best reply to (s^S, α^S) for S in $\Gamma(T, p, U, v, c)$, for any $c > 0$.

Instead, R has a set of strategies and a payoff structure in $\Gamma(T, p, U, v, c)$ that are different from those of game $\Gamma(T, p, U, v, 0)$. In particular, R's strategy set in $\Gamma(T, p, U, v, 0)$ is $\{s2\} \times \mathcal{A}$ which is a restriction of $\{s1, s2\} \times \mathcal{A}$, R's strategy set in $\Gamma(T, p, U, v, c)$; R's payoff structure in $\Gamma(T, p, U, v, c)$ is that faced in $\Gamma(T, p, U, v, 0)$ with the addition, in the light of A5 (fixed positive acquisition cost), of $-c$ in case R chooses $s = s2$. So, R's expected utility in game $\Gamma(T, p, U, v, c)$ under profile $(\mu^S, (s^S, \alpha^S))$ and beliefs $\beta^S(x|\mu, s) = \left(\beta_1^S(x|\mu, s), \ldots, \beta_n^S(x|\mu, s)\right) \in \Delta T$ is given by:

$$\sum_{t \in T} p_t \left(\sum_{k \in T} \beta_k^S(\mu^S(t)|\mu^S, s2)v(t, \mu^S(t), \alpha^S(\mu(t), s2)) - c\right). \tag{2}$$

Consider a deviation by R from (s^S, α^S) to $(s1, \alpha')$, with $\alpha'(x, s1) = \rho^{s1}(\mu^S)$ for all $x \in X$. Note that, by definition of ρ^{s1}, this is the best deviation entailing $s = s1$ that is available to R. Note also that, because R observes no signal under $s1$, R's posterior beliefs are equal to priors p, so that R's expected utility for deviating to $(s1, \alpha')$ is given by:

$$\sum_{t \in T} p_t v(t, \mu^S(t), \rho^{s1}(\mu^S)). \tag{3}$$

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Let us set:

$$\bar{c}\left((\mu^S, (s^S, \alpha^S))\right) = \sum_{t \in T} p_t \left(\sum_{k \in T} \beta_k^S(\mu^S(t)|\mu^S, s2) v(t, \mu^S(t), \alpha^S(\mu(t), s2)) \right) - \sum_{t \in T} p_t v(t, \mu^S(t), \rho^{s1}(\mu^S)).$$

(4)

If the information conveyed by μ^S on sender's types is valuable to R, then there exists $\bar{v}$ such that $\bar{c}\left((\mu^S, (s^S, \alpha^S))\right) \geq \bar{v} > 0$. By construction, $c \leq \bar{c}$ implies that expected utility (2) is not lower than expected utility (3), i.e., deviation $(s1, \alpha')$ is not profitable with respect to (s^S, α^S).

Consider a deviation by R from (s^S, α^S) to $(s2, \alpha'')$. Since $(\mu^S, (s^S, \alpha^S))$ is an equilibrium of $\Gamma(T, p, U, v, 0)$, it follows that α^S is chosen optimally by R given $s^S = s2$ and μ^S. Hence, $(s2, \alpha'')$ cannot be a profitable deviation in the game $\Gamma(T, p, U, v, c)$, for all $c > 0$.

These observations on the possible deviations by R imply that for any $c \leq \bar{c}\left((\mu^S, (s^S, \alpha^S))\right)$ the strategy (s^S, α^S) is a best reply to μ^S for S in game $\Gamma(T, p, U, v, c)$. This, in turn, implies that, for any $c \leq \bar{c}\left((\mu^S, (s^S, \alpha^S))\right)$, $(\mu^S, (s^S, \alpha^S))$ is an equilibrium of the SCAS game $\Gamma(T, p, U, v, c)$. $\qquad\square$

The main insight of Proposition 2 is that, whenever information on sender's types is valuable to R, any separating or semi-separating equilibrium of a SFAS game is also an equilibrium of the SCAS games with same utility functions U and v and acquisition costs sufficiently small. From this it is straightforward to conclude that, given a SFAS game, all separating and semi-separating equilibria in which the sender's profile conveys information on senders types that is valuable to R are also equilibria of any associated SCAS game if acquisition costs are low enough.

5.3   Surplus maximization

So far, we have shown that in a SCAS game the presence of costs to acquire the signal leads to the existence of a unique pooling equilibrium outcome – where all sender's types pool on the null signal – while separating and semi-separating equilibria may exist in the same number and quality of those of the associated SFAS game, but only if acquisition costs are not too large and information on types is valuable to the receiver. Furthermore, as it is shown in Section 6, a no-signal pooling equilibrium turns out to be robust to many standard refinements of equilibria. This suggests that in a SCAS game a no-signal pooling equilibrium is typically more focal than in the associated SFAS game, and this is more likely to be the case the larger the acquisition costs.

Is this good or bad news? From a welfare perspective, a no-signal pooling equilibrium can lead to either a better or worse outcome with respect to an alternative separating or semi-separating equilibrium. Of course, this is true for standard signaling games as well; but in the presence of

costs to acquire the signal a no-signal pooling equilibrium is more likely to be the second best, and the likelihood of this increases in the cost size. The next proposition makes this point clear:

PROPOSITION **3.** *Given a SFAS game* $\Gamma(T, p, U, v, 0)$, *there exists* $\bar{c}^P > 0$ *such that in all SCAS games* $\Gamma(T, p, U, v, c)$ *with* $c \geq \bar{c}^P$ *the pooling equilibrium* $(\mu^P, (s^P, \alpha^P))$ *entails a total surplus which is not lower than in any other equilibrium.*

Proof. Let $\mathcal{E}^S(\Gamma(T, p, U, v, 0)) \subseteq \mathcal{M} \times \{s1, s2\} \times \mathcal{A}$ be the set of separating and semi-separating equilibria of the SFAS game $\Gamma(T, p, U, v, 0)$, i.e., $(\mu, (s, \alpha)) \in \mathcal{E}^S$ if and only if $(\mu^S, (s^S, \alpha^S))$ satisfies E1-E3 and $\mu^S \in M = \{\mu \in \mathcal{M} | \exists t, t' \in T, \mu(t) \neq \mu(t')\}$. Hence, $\mathcal{E}^S$ also contains all separating and semi-separating equilibria of any SCAS game $\Gamma(T, p, U, v, 0)$, $c > 0$.

For any $(\mu, (s, \alpha)) \in \mathcal{E}^S(\Gamma(T, p, U, v, 0))$, the total surplus of the SFAS game $\Gamma(T, p, U, v, 0)$ is given by:

$$TS(\mu, (s, \alpha)) = \sum_{t \in T} p_t U(t, \mu(t), \alpha(\mu(t), s)) + \sum_{t \in T} \beta_t(\mu(t) | \mu, s) v(t, \mu(t), \alpha(\mu(t), s));$$

while the total surplus associated with the pooling equilibrium $(\mu^P, (s^P, \alpha^P))$ of both the SFAS game $\Gamma(T, p, U, v, 0)$ and the SCAS game $\Gamma(T, p, U, v, c)$ is given by:

$$TS(\mu^P, (s^P, \alpha^P)) = \sum_{t \in T} p_t U(t, 0, \rho^{s1}(\mu^P)) + \sum_{t \in T} p_t v(t, 0, \rho^{s1}(\mu^P)).$$

Let $\bar{c}^P$ be defined as follows:

$$\bar{c}^P = \begin{cases} \left(\displaystyle\sup_{(\mu, (s, \mu)) \in \mathcal{E}(\Gamma(T, p, U, v, 0))} TS(\mu, (s, \alpha)) \right) - TS(\mu^P, (s^P, \alpha^P)) & \text{if greater than } 0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

Consider now the SCAS games $\Gamma(T, p, U, v, c')$ such that $c' \geq \bar{c}^P$. By Proposition 1, $(\mu^P, (s^P, \alpha^P))$ leads to the unique pooling equilibrium outcome, and therefore $TS(\mu^P, (s^P, \alpha^P))$ is trivially the maximum total surplus among pooling equilibria.

If no separating or semi-separating equilibrium $(\mu^S, (s^S, \alpha^S)) \in \mathcal{E}^S(\Gamma(T, p, U, v, 0))$ is an equilibrium of $\Gamma(T, p, U, v, c')$, then $(\mu^P, (s^P, \alpha^P))$ is trivially a surplus maximizer in $\mathcal{E}^S(\Gamma(T, p, U, v, c'))$. If there exists a separating or semi-separating equilibrium $(\mu^S, (s^S, \alpha^S)) \in \mathcal{E}^S(\Gamma(T, p, U, v, 0))$ that also belongs to $\mathcal{E}^S(\Gamma(T, p, U, v, c'))$, then $(\mu^S, (s^S, \alpha^S))$ entails a total surplus equal to $TS(\mu^S, (s^S, \alpha^S)) - c'$, because of assumption A5 (fixed positive acquisition cost). Hence, we can conclude that, if $c' \geq \bar{c}^P$, then $TS(\mu^P, (s^P, \alpha^P)) \geq TS(\mu, (s, \alpha)) - c'$ for all $(\mu, (s, \alpha)) \in \mathcal{E}^S(\Gamma(T, p, U, v, 0))$. $\square$

A straightforward implication of Proposition 3 is that whenever a no-signal pooling equilibrium is desirable in a standard signaling setup, then it is a fortiori desirable in the presence of acquisition

costs, no matter how big or small they are. Of course, the converse does not hold as a separating equilibrium could well be the desirable outcome in a SFAS game, but it may not be viable in the associated SCAS game with sufficiently large acquisition costs. The following corollary formalizes:

COROLLARY 1. *If $(\mu^P, (s^P, \alpha^P))$ is a pooling equilibrium of the SFAS game $\Gamma(T, p, U, v, 0)$ and entails a total surplus which is not lower than in any other equilibrium, then $(\mu^P, (s^P, \alpha^P))$ also entails this for all SCAS games $\Gamma(T, p, U, v, c)$. Moreover, the converse does not hold.*

Proof. The first claim follows directly from the last observation of the proof of Proposition 3. Indeed, if $TS(\mu^P, (s^P, \alpha^P))$ is the maximum equilibrium total surplus in $\Gamma(T, p, U, v, 0)$, then $\bar{c}^P = 0$, and so $TS(\mu^P, (s^P, \alpha^P))$ is also the maximum equilibrium total surplus in $\Gamma(T, p, U, v, c)$, for any $c > 0$. That the converse does not hold follows from the observation that $TS(\mu^P, (s^P, \alpha^P))$ is not always the maximum total surplus in a SFAS game $\Gamma(T, p, U, v, 0)$, while it must be so for the associated SCAS game $\Gamma(T, p, U, v, c')$ for c' sufficiently high, e.g., for $c' > \max_{(\mu,(s,\mu))\in\mathcal{E}(\Gamma(T,p,U,v,0))} \bar{c}(\mu, (s, \mu))$.
□

6 On the robustness of the pooling outcome

6.1 Refinements acting on out-of-equilibrium beliefs

Many refinements of Bayes-Nash equilibria have been proposed, especially for signaling games. Most of them follow the idea that out-of-equilibrium beliefs should not be totally free, but need to satisfy some criterion of reasonableness. All such refinements imply sequentiality (Kreps and Wilson, 1982).

Cho and Kreps (1987) introduce the Intuitive Criterion which requires that out-of-equilibrium beliefs place zero weight on types that can never gain from deviating from the considered equilibrium.

Banks and Sobel (1987) introduce Divinity which requires that out-of-equilibrium beliefs place relatively more weight on types that gain more from deviating from the considered equilibrium. They also introduce Universal Divinity which requires that beliefs survives Divinity for all possible priors. The surviving beliefs do not depend on the priors – while those surviving Divinity in general do.

Motivated by Banks and Sobel (1987), Cho and Kreps (1987) have also introduced D1, which requires that out-of-equilibrium beliefs are supported on types that have the most to gain from deviating from the considered equilibrium, and D2, which requires to place zero weight on types that always have some other type gaining strictly from deviating from the considered equilibrium.

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In general Divinity turns out to be a weakening of D1, while Universal Divinity to be a strengthening of D2.

Cho and Sobel (1990) demonstrate that, for monotonic signaling games, the set of D1 and Universal Divinity are equivalent to Strategic Stability (Kohlberg and Mertens, 1986); moreover, if the single-crossing property is satisfied, then D1 yields a unique equilibrium.

The perfect sequential equilibrium by Grossman and Perry (1986) is more tricky. It selects equilibria that survive backward induction in a game where nodes are not only identified by paths of play but also by beliefs at such nodes – they call a strategy of this game a metastrategy. The concept of perfect sequential equilibrium selects a set of equilibria – possibly empty – that is a subset of that obtained with the intuitive criterion.

The undefeated equilibrium by Mailath et al. (1993) rests on totally different grounds and restrict beliefs according to payoff comparisons at distinct sequential equilibria. More precisely, a first sequential equilibrium is defeated by a second sequential equilibrium if there exists a non-negligible set of types that prefer to deviate from what they do in the first equilibrium to what they do in the second equilibrium and, at the same time, the beliefs of the non-deviating types in the first equilibrium are not consistent with such a deviation for this set of types. A sequential equilibrium is undefeated if no other equilibrium defeats it.

These refinements relate to each other in a non trivial way,[9] but all rely on the possibility that a deviation by the sender triggers a path of play along which the receiver gets some piece of information that is unexpected along the equilibrium path. However, in the pooling equilibrium of a SCAS game the receiver does not acquire the signal, so that this possibility does not exist. Intuitively, this is why refinements based on out-of-equilibrium beliefs do not have a bite in such case.

To formalize this point let us introduce the following definitions. For given strategy profile $(\mu, (s, \alpha))$ and priors p, R has beliefs $\beta(x|\mu, s) \in \Delta T$ associated with each of her information sets where an action in Y has to be chosen. Denote with $X^e(\mu, (s, \alpha)) \subseteq X$ the set of signals that R can observe on information sets along the equilibrium path, i.e., at information sets that contain decision nodes along the equilibrium path. Denote with $X^o(\mu, (s, \alpha)) = X \setminus X^e(\mu, (s, \alpha))$ the set of signals that R can observe only at information sets off the equilibrium path, i.e., at information sets that do not contain decision nodes lying on the equilibrium path.

Moreover, denote with $X^{o1}(\mu, (s, \alpha)) \subseteq X^o(\mu, (s, \alpha))$ the set of signals off the equilibrium path that R cannot observe as a consequence of S deviating from μ because a deviation by R is required.

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[9]In particular, an equilibrium satisfying D1 must also satisfy D2 which in turn requires to satisfy the Intuitive Criterion; an equilibrium satisfying Universal Divinity must also satisfy Divinity which in turn requires to satisfy the Intuitive Criterion; an Undefeated equilibrium need only be Sequential, while a Perfect Sequential Equilibrium must satisfy the Intuitive Criterion.

Also, denote with $X^{o2}(\mu, (s, \alpha)) = X^o(\mu, (s, \alpha)) \setminus X^{o1}(\mu, (s, \alpha))$ the set of signals off the equilibrium path that R can potentially observe as a consequence of S deviating from μ. In a SCAS game, we call *receiver-triggered out-of-equilibrium beliefs* the beliefs held by R which are activated by signals in $X^{o1}(\mu, (s, \alpha))$,[10] and we call *sender-triggered out-of-equilibrium beliefs* the beliefs held by R which are activated by signals in $X^{o2}(\mu, (s, \alpha))$.

A refinement that rules away equilibria by restricting admissible beliefs to a subset of those possibly activated by signals in $X^o(\mu, (s, \alpha))$ can be regarded as a *refinement acting on out-of-equilibrium beliefs*. All equilibrium refinements discussed above are evidently refinements acting on out-of-equilibrium beliefs. Note that, although such refinements require R to observe an unexpected signal, they can potentially act on beliefs activated by all $x \in X^o(\mu, (s, \alpha))$, i.e., they act not only on sender-triggered out-of-equilibrium beliefs, but also on receiver-triggered ones.

The following proposition establishes that the pooling outcome identified by Proposition 1 is robust to any refinement acting on out-of-equilibrium beliefs:

PROPOSITION 4. *In the SCAS game, the profile $(\mu^P, (s^P, \alpha^P))$ where $\mu^P(t) = 0$ for all $t \in T$, $s^P = s1$, and $\alpha^P(x, s1) = \alpha^P(x, s2) = \rho^{s1}(\mu^P)$ is an equilibrium that survives any possible equilibrium refinement acting on out-of-equilibrium beliefs.*

Proof. The considered equilibrium profile $(\mu^P, (s^P, \alpha^P))$ prescribes that R plays $s^P = s1$, so it follows that $X^e(\mu^P, (s^P, \alpha^P)) = X^{o2}(\mu^P, (s^P, \alpha^P)) = \emptyset$ and $X^{o1}(\mu^P, (s^P, \alpha^P)) = X$, because R can observe a signal $x \in X$ only if she deviates from her equilibrium strategy. In particular, no sender-triggered out-of-equilibrium belief exists because, since $s^P = s_1$, any $x \in X$ that is chosen by S leads to the same R's information set, which is on the equilibrium path. Hence, at this information set, R must have constant beliefs which are identical to the priors p and which cannot be refined away by refinements acting on out-of-equilibrium beliefs.

So, refinements acting on out-of-equilibrium beliefs can rule out only beliefs associated with information sets that become active when a signal in $x \in X^{o1}(\mu^P, (s^P, \alpha^P)) = X$ is observed. However, none of these receiver-triggered out-of-equilibrium beliefs is necessary to sustain the considered equilibrium. To see why, note that R always uses the priors p and S's strategy μ^P to evaluate whether to deviate or not from (s^P, α^P); so, there is no deviation by S that can induce R to deviate from (α^P, s^P), as (α^P, s^P) is a best response to μ^P given p, no matter what are the receiver-triggered out-of-equilibrium beliefs held by R; since also μ^P is a best response to (α^P, s^P), it follows that S has no strictly profitable deviation from μ^P, and this is again independent of the receiver-triggered out-of-equilibrium beliefs held by R. Hence, no refinement acting on out-of-equilibrium beliefs can refine away the considered equilibrium $(\mu^P, (s^P, \alpha^P))$. $\square$

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[10]We note that, although all receiver-triggered beliefs require a deviation by R to be activated, some of them may additionally require a previous deviation by S to be activated.

Let us remark that refinements acting on out-of-equilibrium beliefs do have a bite in a SCAS game. Actually, they do refine away a lot of separating equilibria, in a manner similar to what they do in a signaling game without costly acquisition of signals. Indeed, in any separating equilibrium with profile $(\mu^S, (s^S, \alpha^S))$, the receiver R must be playing $s^S = s2$, so that signals $x \in X^{o2}(\mu^S, (s^S, \alpha^S)) = X \setminus X^e(\mu^S, (s^S, \alpha^S)) \neq \emptyset$ lead to sender-triggered out-of-equilibrium beliefs which crucially sustain separation by punishing S's deviations from μ^S – as it happens in standard signaling games.

6.2   Inviting to acquire the signal through further signaling

It seems natural to ask whether the prominence of separation is restored if S has the possibility to communicate to R that he is actually sending an informative signal – i.e., a signal that separates (at least partly) types – and that therefore the signal is worth acquisition.

One can think of many situations where indeed the sender can send, together with the main signal x, an accompanying costly signal, say z, that acts as an invitation for the receiver to engage in the costly acquisition of x. We show that, in fact, not much can be restored by the use of z.

To have an intuition of why it is so, note that in order for the accompanying signal z to help separation, types must separate on z. Indeed, if separation is attained on x and the receiver acquires x, then all types would strictly prefer to save on costs and pool on a null z. So, suppose that separation is effectively attained on z. Then, the receiver strictly prefers not to incur the cost of acquiring x – since its acquisition would add no useful information – with the result that the only communication that takes place is that through z. However, in order for this kind of separation to be more robust than a pooling equilibrium, it is *necessary* that the sender's utility function satisfies an equivalent of the single-crossing property on types and z – e.g., by satisfying the single-crossing property on x and z so that, for separating profiles, the single-crossing on types and x induces the single-crossing on types and z – but this is not guaranteed in general. At any rate, even if such a necessary condition holds, to restore the prominence of separation the receiver must be able to acquire z for free. In fact, if the receiver has to incur a positive cost to acquire z – no matter how small – then the strategic complementarity (between signaling and acquiring the signal) is still in place and the very same arguments discussed in Subsection 6.1 apply also in this setup.

To provide a more formal discussion of these ideas, we construct a variant of the SCAS game that we call SCAS *game with invitation signal* (SCAS-IS), and for which we show an equivalent of Proposition 1 and Proposition 4. Since both the details of the SCAS-IS game and the proof are rather long and add little to intuition, we provide them in Appendix A.

6.3 Smooth acquisition costs

One can think of the process of signal acquisition as a smooth one: the greater the cost incurred to acquire the signal, the greater the acquisition of the signal content. In this respect, a natural question to ask is whether separation of sender's types becomes more likely under such a smooth process. It turns out that the pooling outcome retains its focality, as the robustness of a no-signal pooling equilibrium does not depend at all on the fact that R's acquisition choice is binary, i.e., either pay the acquisition cost and acquire x or pay nothing and acquire nothing.

A simple way to model a smooth process of signal acquisition is to consider a stochastic acquisition where the probability of acquiring x is an increasing function of the cost paid.[11] Suppose R has the possibility to choose a level of acquisition effort $e \in [0, 1]$, which replaces the choice of $s \in \{s1, s2\}$; also, with probability $1 - e$ no signal is acquired, while with probability e the signal is acquired. Note that a no-signal pooling equilibrium is still sustained by a form of strategic complementarity: if S chooses $x = 0$ for all $t \in T$ then R's optimal choice is $e = 0$ (so never acquiring the signal), and if R chooses $e = 0$ then S's optimal choice is $x = 0$ for all $t \in T$.

Similarly to what done for the SCAS-IS game, in order to provide a formal argument in support of intuition we provide a variant of the SCAS game that accommodates the idea of smooth acquisition costs. We call this variant the SCAS *game with acquisition effort* (SCAS-AE); for such a class of games we show an equivalent of Proposition 1 and Proposition 4. Since also in this case both the details of the game and the proof are rather long and add little to the intuition described in the current subsection, we put them in Appendix B.

6.4 Signal not purely costly to the Sender

The SCAS game studied in this paper accommodates cases where the signal x is purely dissipative – it is always a net cost for S and of no intrinsic utility (or some disutility) for R – as well as cases where the signal x is of some intrinsic value to the receiver. However, the model does not accommodate the case where x is not a pure net cost for the sender. Indeed, this case is ruled out by assumption A3.

In particular, assumption A3 together with the fact that X has a lower bound at 0 implies that all sender's types strictly prefer, other things being equal, to set $x = 0$. Formally, we have that A3 and $X = \mathbb{R}_+$ imply that:

$$x^*(t, y) \equiv \arg\max_{x \in X} U(t, x, y) = 0 , \quad \text{for all } t \in T, y \in Y. \tag{5}$$

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[11]A different way to model smooth process of signal acquisition is to have the signal x always acquired but with some blurring noise whose incidence negatively depends on e.

It turns out that the kind of strategic complementarity that supports the pooling outcome in a SCAS game may not exists if (5) does not hold. However, we stress that what is crucial to our results in (5) is not that 0 is the common best signal for all sender's types in the absence of a signaling value – an assumption which, in fact, can easily be substituted with a common optimal $x^* > 0$ for all $t \in T$; what really matters for the existence of the needed strategic complementarity is that a common best signal exists for all types. To see why, consider the extreme case where $x^*(t, y)$ is one-to-one in t for any given y. This means that, even in the case that R chooses $s = s1$, all sender's types would find it optimal to choose a different x. If the information about the sender's type is sufficiently valuable to R, it becomes impossible for a profile with no signal acquisition to be an equilibrium. Indeed, since types separate independently of R's behavior, to rule out equilibria where the signal is not acquired is sufficient to have that the expected value for R of discovering S's type is greater than the cost of acquiring the signal.

Let us conclude with a few remarks that, in our opinion, indicate that acquisition costs – and in general the analysis conducted in this paper – might be relevant even when A3 does not hold.

One remark regards the refinement potential of arbitrarily small acquisition costs in a standard signaling game. Note that if $x^*(t, y)$ is one-to-one in t for any given y, then the incentive for S's types to separate does not come from the fact that R acquires the signal, but from the fact that each type has its own preferred x. This rules out all pooling equilibria in a SCAS game, but it does not so in the associated SFAS game (i.e., in a standard signaling game). In fact, in a SFAS game R always chooses $s = s2$, and in particular it does so also when all S's types pool on the same $\bar{x}$; this allows for out-of-equilibrium beliefs on the part of R that harshly punish types who deviate from $\bar{x}$, sustaining the pooling equilibrium. In a SCAS game, instead, R would switch from $s = s2$ to $s = s1$, leaving each type $t \in T$ free to switch to his preferred $x^*(t, y)$. Perhaps interestingly, this argument shows that an arbitrarily small acquisition cost rules out all pooling equilibria in signaling games where types strictly prefer different signal levels.

Another remark regards the potential backfiring of mandatory disclosure policies. Consider a SCAS game where x represents costly disclosure of some characteristic on the part of the sender, and suppose that a public authority wants to keep x above a certain threshold. If $x^*(t, y)$ is one-to-one in t for the relevant range of y, then some disclosure will certainly happen as no pooling can be sustained in equilibrium. However, if the public authority imposes a minimum $\bar{x}$, then it can happen that separation collapses and a pooling equilibrium on $\bar{x}$ with no signal acquisition emerges. In particular this will happen whenever $\bar{x} \geq \max_{t \in T} x^*(t, \bar{y})$, where $\bar{y}$ is the best action for R under $s1$ when all types pool on $\bar{x}$ (for smaller values of $\bar{x}$ a partial pooling can emerge, instead). This may lead to a loss in terms of information transmission that more than offsets the targeted benefits of a high x.

6.5 Publicly committing to acquire the signal

In the SCAS game studied in Section 4 and 5, as well as in the variants discussed in Subsection 6.2, 6.3, and 6.4, there is no possibility for the Sender, prior to choosing the signal, to observe the choice of the Receiver between acquiring and not acquiring the signal. This neglected case is strategically equivalent to a situation where the receiver R must publicly commit to choose $s1$ or $s2$, so that S learns if R will acquire the signal *before* choosing what signal to send. It is not too complicated to see that if R must publicly commit to a given $s \in \{s1, s2\}$, and information is sufficiently valuable to her, then the prominence of separation is restored.

Consider a variant of the SCAS game $\Gamma(T, p, U, v, c)$ where R must[12] initially commit herself to choose $\bar{s} \in \{s1, s2\}$ and suppose also that such commitment is observed by S before he choses whatsignal to send. This configures an additional initial stage of the game where R announces $\bar{s}$, followed by a second stage of the game where S chooses the signal x, and then a third stage where R plays $\bar{s}$ and chooses an action y. In this setup, S can condition the choice of the signal on $\bar{s}$, so that his strategy is now represented by function $\bar{\mu} : T \times \{s1, s2\} \to X$. Note that this setup configures two distinct subgames: one signaling (sub)game in which R has committed to $s1$ and S knows that his signal will never be acquired by R (basically, signalling is impossibile), and another signaling (sub)game where R has committed to $s2$ and S knows that his signal will always be acquired (basically, the associated SFAS game). Such dynamic structure of the game naturally calls for an equilibrium concept that entails backward reasoning. A simple way to do so is to look for subgame perfection after having refined the equilibria of each signaling subgame using standard refinements acting on out-of-equilibrium beliefs.

In the signaling subgame where R has committed to $s1$, there is just one Bayes-Nash equilibrium: all types of S pool on $x = 0$, since $x > 0$ is costly and S is certain that R will never observe x. Denote S's strategyin this subgame with μ^P, with $\mu^P(t) = 0$ for all t. In the signaling subgame where R has committed to $s2$, there are many Bayes-Nash equilibria, both pooling and separating. Since the cost c of choosing $s2$ is sunk, in this subgame there are exactly the same equilibria of the associated SFAS game $\Gamma(T, p, U, v, 0)$, i.e., they are the same of a standard signaling game. Widely used refinements acting of out-of-equilibrium beliefs (such as, e.g., D1) refine away all pooling equilibria and most separating equilibria. For the sake of the argument consider the typical case where only the best separating equilibrium survives, i.e., the equilibrium where all sender types separate and each type spends on the signal x the minimum necessary to do so. Denote S's strategy in this equilibrium with μ^S. So, S's strategy in the full game can be written as $\bar{\mu} = (\mu^P, \mu^S)$.

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[12]We stress that if R is not forced to commit, but has just the option to do so, the main thrust of the argument still applies. The reason is that, as long as information is valuable to R, the option to commit to $s2$ will always be exercised.

Using backward reasoning, R anticipates that by committing to $s1$ she will end up in a pooling equilibrium where she plays $y = \rho^{s1}(\mu^P)$, while by committing to $s2$ she will end up in a fully separating equilibrium where she plays, for each t, $y(t) \in \arg\max_{y \in Y} u(t, \mu^S(t), y)$. If the information conveyed by μ^S is valuable to R in the SFAS game $\Gamma(T, p, U, v, 0)$, then it is straightforward to see that for a positive acquisition cost c which is small enough, R strictly prefers to end up in the fully separating equilibrium, and therefore she will commit to $s2$. In such a case, the prominence of the outcome of full separation is restored.

Let us end this discussion with a more general point regarding the actions that the receiver can take to facilitate communication. To the extent that information transmission is valuable to the receiver, it is reasonable to expect that the receiver acts (and even incurs costs) in order to facilitate the emergence of a fully separating equilibrium. If public commitment to the acquisition of the signal is possibile or if the acquisition can be made before the signal is sent, then separation is actually the most likely outcome. But how likely it is that this is the case? For the signal to be effectively acquired *before* the sender sends it, we must be in a situation where the sender and the receiver communicate through a channel that the receiver can "switch on" at her will and whose on/off status is easily observable (i.e., at no cost) by the sender. In addition, as the cost of keeping the channel switched on reasonbly depends on how long it is left in such a state, another crucial requirement is that the receiver must know approximately *when* the sender is going to send the signal, otherwise another coordination problem arises. While this setup certainly fits some real cases of signaling, it hardly fits most of them.

A case that is perhaps more frequent in real interactions is the one where the receiver can commit to acquire the signal and is also able to inform the sender of the commitment *before* the sender decides what signal to send. When signal acquisition takes the form of buying the necessary "hearing" tools, or when it takes the form of moving to the correct "listening" location, then public commitment sounds reasonable. However, if signal acquisition is a matter of cognitive effort or attention, then the possibility of public commitment seems far less likely, especially for what concerns letting the sender know about the commitment. Summing up, it seems safe to say that the pooling outcome is not robust to the possibility for the receiver to publicly commit to acquire the signal, but such a possibility is far from being the rule.

Acknowledgements

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A The SCAS game with Invitation Signal (SCAS-IS)

Consider a SCAS game and add a preliminary stage where S can send an additional costly signal $z \in Z = \mathbb{R}_+$. Note that, if z can be used by S to signal to R in a credible way that the signal x is informative, then z can indeed be used by S to invite R to acquire the signal x. This kind of credibility is assumed by means of a single-crossing property over x and z, for all $t \in T$, which captures the fact that the types who are investing resources to signal through x have relatively smaller costs for sending z (see assumption B5).

As it happens for the signal x, also the invitation signal z can be acquired or not by R. The choice by R to acquire z is denoted with $r \in \{r1, r2\}$, where $r = r1$ means that R does not acquires z while $r = r2$ means that R acquires z.

In a SCAS-IS game, utility for S is $U : T \times X \times Z \times Y \to \mathbb{R}$, and utility for R is $V : T \times X \times Y \times \{s1, s2\} \times \{r1, r2\} \to \mathbb{R}$. The following assumptions on utility functions hold:

B1. *continuity:* U and V are continuous over x, z, and y;

B2. *monotonicity in action:* U is strictly increasing in y;

B3. *costly signaling:* U is strictly decreasing in x and z;

B4. *single-crossing property on (t, x):* $U(t, x, z, y) \leq U(t, x', z, y')$, with $x' > x$, implies that $U(t', x, z, y) < U(t', x', z, y')$ for all $t' > t$, $y, y' \in Y$, and $z \in Z$;

B5. *single-crossing property on (x, z):* $U(t, x, z, y) \leq U(t, x, z', y')$, with $z' > z$, implies that $U(t, x', z, y) < U(t, x', z', y')$ for all $x' > x$, $y, y' \in Y$, and $t \in T$;

B6. *fixed positive cost of acquiring signal x:* $V(t, x, y, s1, r) - V(t, x, y, s2, r) = c_x > 0$ for all $t \in T$, $x \in X$, $y \in Y$, $r \in \{r1, r2\}$;

B7. *fixed positive cost of acquiring signal z:* $V(t, x, y, s, r1) - V(t, x, y, s, r2) = c_z > 0$ for all $t \in T$, $x \in X$, $y \in Y$, $s \in \{s1, s2\}$;

In the light of B6 and B7, we have that $v(t, x, y) + c_x + c_z = V(t, x, y, s2, r2)$, $v(t, x, y) + c_x = V(t, x, y, s2, r1)$, and $v(t, x, y) + c_z = V(t, x, y, s1, r2)$.

In a SCAS-IS game, a strategy for S is a pair (ζ, μ) where $\zeta : T \to Z$ describes a type's choice of z while $\mu \in \mathcal{M}$ describes, as in a SCAS game, a type's choice of x; we denote with $\mathcal{Z}$ the set of all possible ζ. A strategy for R is a triple (r, σ, α) where:

- $r \in \{r1, r2\}$ describes R's choice to acquire the invitation signal z;

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- $\sigma : Z \times \{r1, r2\} \to \{s1, s2\}$ describes R's choice to acquire the signal x conditional on z, with σ satisfying $\sigma(z, r1) = \sigma(z', r1)$ for all $z, z' \in Z$, i.e., with the choice of s being unconditional on z if $r = r1$; we denote with Σ the set of all such functions;

- $\alpha : X \times \{s1, s2\} \times \{r1, r2\} \times Z \to Y$ describes (with a slight abuse of notation) R's choice of y, with α satisfying:

    - $\alpha(x, s1, r2, z) = \alpha(x', s1, r2, z)$ for all $x, x' \in X$,
    - $\alpha(x, s1, r1, z) = \alpha(x', s1, r1, z')$ for all $x, x' \in X$ and $z, z' \in Z$,
    - $\alpha(x, s2, r1, z) = \alpha(x, s2, r1, z')$ for all $z, z' \in Z$,

    i.e., with the choice of y being unconditional on z if $r = r1$ and unconditional on x if $s = s1$; we denote with $\mathcal{A}^{IS}$ the set of all such functions.

For given (ζ, μ) and (r, σ, α), R has posterior beliefs that depend on both her choice of r and her choice of s. If R chooses both $s = s2$ and $r = r2$, then she has posterior beliefs $\beta(z, x|(\zeta, \mu), s2, r2)$ $= (\beta_t(z, x|(\zeta, \mu), s2, r2), \ldots, \beta_n(z, x|(\zeta, \mu), s2, r2)) \in \Delta T$ where $\beta_t(z, x|(\zeta, \mu), s2, r2)$ denotes the probability that S is of type t, conditional on the observation of z and x. If R chooses $s = s1$ and $r = r2$ then she does not observe x but still observes z, so that her posteriors are given by $\beta_t(z, x|(\zeta, \mu), s1, r2) = \beta_t(z, x'|(\zeta, \mu), s1, r2)$, for all $t \in T$ and all $x, x' \in X$. If R chooses $s = s2$ and $r = r1$ then she observes x but does not observe z, so that her posteriors are given by $\beta_t(z, x|(\zeta, \mu), s2, r1) = \beta_t(z', x|(\zeta, \mu), s2, r1)$, for all $t \in T$ and all $z, z' \in Z$. All these beliefs can be obtained by Bayes's rule, if applicable, or be chosen otherwise. Finally, if R chooses $s = s1$ and $r = r1$ then she can only rely on her priors – no new information is acquired – so that posteriors are trivially identical to priors: $\beta_t(z, x|(\zeta, \mu), s1, r1) = \beta_t(z', x'|(\zeta, \mu), s1, r1) = p_t$, for all $t \in T$ and all $x, x' \in X$ and $z, z' \in Z$.

We also introduce an equivalent of assumption A6 for the current setup, which accommodates the fact that R obtains no information if she refuses to acquire both z and x:

B8. *uniqueness of best action under s1 and r1:*
   $\rho^{s1, r1}(\mu) = \arg\max_{y \in Y} \sum_{t \in T} p_t v(t, \mu(t), y)$ exists and is single valued.

Note that, since z does not affect R's utility, function $\rho^{s1, r1}(\mu)$ is the same of function $\rho^{s1}(\mu)$ of a SCAS game.

To define the equilibrium of a SCAS-IS game in a compact and readable form, let us introduce

some further notation:

$$\mathbb{E}V[s,\alpha|r1,(\zeta,\mu)] = \sum_{t\in T} p_t \left(\sum_{k\in T} \beta_k(\zeta(t),\mu(t)|(\zeta,\mu),s,r1)V(\mu(t),\alpha(\mu(t),s,r,\zeta(t)),s,r1) \right);$$

$$\mathbb{E}V[\sigma,\alpha|r2,(\zeta,\mu)] = \sum_{t\in T} p_t \left(\sum_{k\in T} \beta_k(\zeta(t),\mu(t)|(\zeta,\mu),\sigma(\zeta(t),r2),r2)\cdot \right.$$
$$\left. \cdot V(\mu(t),\alpha(\mu(t),\sigma(\zeta(t),r2),r2,\zeta(t)),\sigma(\zeta(t),r2),r2)) \right);$$

where $\mathbb{E}V[s,\alpha|r1,(\zeta,\mu)]$ is R's expected utility against (ζ,μ) when choosing $r=r1$, $\sigma(z,r)=s$ for all $z\in Z$, and some $\alpha\in\mathcal{A}^{IS}$, while $\mathbb{E}V[\sigma,\alpha|r2,(\zeta,\mu)]$ is R's expected utility against (ζ,μ) when choosing $r=r2$, some $\sigma\in\Sigma$, and some $\alpha\in\mathcal{A}^{IS}$.

DEFINITION **2.** *(weak Perfect Bayes-Nash equilibrium of the SCAS-IS game)*
A wPBE equilibrium of a SCAS-IS game is a profile of strategies $((\zeta,\mu),(r,\sigma,\alpha))$ such that:

F1. $(\zeta(t),\mu(t)) \in \underset{z\in Z,x\in X}{\arg\max}\ U(t,x,z,\alpha(x,\sigma(z,r),r,z))$, *for all $t\in T$;*

F2. *for all $x\in X$ and $z\in Z$, there exists beliefs $\beta(z,x|(\zeta,\mu),s,r)\in\Delta T$ such that (r,σ,α) satisfies:*

F2.1. $\alpha(x,s1,r1,z) = \rho^{s1,r1}(\mu)$ *for all $x\in X$ and $z\in Z$;*

F2.2. $\alpha(x,s1,r2,z) \in \underset{y\in Y}{\arg\max}\sum_{t\in T} \beta_t(z,x|(\zeta,\mu),s1,r2)v(t,x,y) - c_z$ *for all $x\in X$ and $z\in Z$;*

F2.3. $\alpha(x,s2,r1,z) \in \underset{y\in Y}{\arg\max}\sum_{t\in T} \beta_t(z,x|(\zeta,\mu),s2,r1)v(t,x,y) - c_x$ *for all $x\in X$ and $z\in Z$;*

F2.4. $\alpha(x,s2,r2,z) \in \underset{y\in Y}{\arg\max}\sum_{t\in T} \beta_t(z,x|(\zeta,\mu),s2,r1)v(t,x,y) - c_x - c_z$ *for all $x\in X$ and $z\in Z$;*

F2.5. $\sigma(z,r1) \in \underset{s\in\{s1,s2\}}{\arg\max}\ \mathbb{E}V[s,\alpha|r1,(\zeta,\mu)]$ *for all $z\in Z$;*

F2.6. $\sigma\in\underset{\sigma\in\Sigma}{\arg\max}\ \mathbb{E}V[\sigma,\alpha|r2,(\zeta,\mu)]$;

F2.7. $r=r1$ *implies that $\mathbb{E}V[\sigma(z,r1),\alpha|r1,(\zeta,\mu)] \geq \mathbb{E}V[\sigma,\alpha|r2,(\zeta,\mu)]$ for all $z\in Z$;*

F2.8. $r=r2$ *implies that $\mathbb{E}V[\sigma(z,r1),\alpha|r1,(\zeta,\mu)] \leq \mathbb{E}V[\sigma,\alpha|r2,(\zeta,\mu)]$ for all $z\in Z$;*

F3. *the beliefs $\beta(z,x|(\zeta,\mu),s,r)\in\Delta T$ are calculated by means of Bayes's rule whenever possible.*

The meaning of F1 is that S must be best-replying to R, while the meaning of F2 is that R must be best-replying to S given her beliefs – this is better seen by noting that, for $r=r1$, condition F2 is substantially identical to E2 for a SCAS game. Condition F3 is straightforward. For the sake of completeness we observe that, in a SCAS-IS game, posterior beliefs along the equilibrium path are the following:

- if $r = r2$ and $s = s2$ then, for all $t \in T$ and for all (z, x) such that $(\zeta(t'), \mu(t')) = (z, x)$ for some $t' \in T$:

$$\beta_t(z, x | ((\zeta, \mu), s, r) = \begin{cases} \dfrac{p_t}{\sum_{k:(\zeta(k), \mu(k)) = (z, x)} p_k} & \text{if } (\zeta(t), \mu(t)) = (z, x) \\ 0 & \text{if } (\zeta(t), \mu(t)) \neq (z, x); \end{cases}$$

- if $r = r1$ and $s = s2$ then, for all $t \in T$ and for all x such that $\mu(t') = x$ for some $t' \in T$:

$$\beta_t(z, x | ((\zeta, \mu), s, r) = \begin{cases} \dfrac{p_t}{\sum_{k:\mu(k) = x} p_k} & \text{if } \mu(t) = x \\ 0 & \text{if } \mu(t) \neq x; \end{cases}$$

- if $r = r2$ and $s = s1$ then, for all $t \in T$ and for all z such that $\zeta(t') = z$ for some $t' \in T$:

$$\beta_t(z, x | ((\zeta, \mu), s, r) = \begin{cases} \dfrac{p_t}{\sum_{k:\zeta(k) = z} p_k} & \text{if } \zeta(t) = z \\ 0 & \text{if } \zeta(t) \neq z; \end{cases}$$

- if $r = r1$ and $s = s1$ then $\beta_t(z, x | (\zeta, \mu), s, r) = p_t$ for all $t \in T$ and for all $x \in X$.

For comparability purposes, we observe that for any given SCAS game $\Gamma(T, p, U, v, c)$ we have, besides the associated SFAS game $\Gamma(T, p, U, v, 0)$, also an associated SCAS-IS game, that we denote with $\Gamma(T, p, \hat{U}, v, c_x, c_z)$ where $c_x = c$ and $\hat{U}(t, x, z, y) = U(t, x, y) + \gamma(z)$ for some appropriate function γ.

We now turn our attention to the relevant beliefs in a SCAS-IS game. For a given strategy profile $((\zeta, \mu), (r, \sigma, \alpha))$ and priors p, R has beliefs $\beta(z, x | (\zeta, \mu), s, r) \in \Delta T$ associated with each of her information sets where an action in Y has to be chosen. Let $\mathbb{S} = Z \times X \cup Z \times \emptyset \cup \emptyset \times X$ be the set of potentially observable pairs of signals – we consider the union with $Z \times \emptyset \cup \emptyset \times X$ to encompass the case where only either z or x is acquired by R. Denote with $(Z, X)^e((\zeta, \mu), (r, \sigma, \alpha)) \subseteq \mathbb{S}$ the set of pairs of signals that R can observe on information sets along the equilibrium path, i.e., at information sets that contain decision nodes along the equilibrium path. Denote with $(Z, X)^o((\zeta, \mu), (r, \sigma, \alpha)) = \mathbb{S} \setminus (Z, X)^e((\zeta, \mu), (r, \sigma, \alpha))$ the set of pairs of signals that R can observe only at information sets off the equilibrium path, i.e., at information sets that do not contain decision nodes lying on the equilibrium path.

Moreover, denote with $(Z, X)^{o1}((\zeta, \mu), (r, \sigma, \alpha)) \subseteq (Z, X)^o((\zeta, \mu), (r, \sigma, \alpha))$ the set of pairs of signals off the equilibrium path that R cannot observe as a consequence of S deviating from (ζ, μ) because a deviation by R is required. Also, denote with $(Z, X)^{o2}((\zeta, \mu), (r, \sigma, \alpha)) = (Z, X)^o((\zeta, \mu), (r, \sigma, \alpha)) \setminus (Z, X)^{o1}((\zeta, \mu), (r, \sigma, \alpha))$ the set of pairs of signals off the equilibrium path that R can potentially observe as a consequence of S deviating from (ζ, μ). Similarly for what done for a SCAS game, we

call *receiver-triggered out-of-equilibrium beliefs* the beliefs held by R which are activated by signals in $(Z,X)^{o1}((\zeta,\mu),(r,\sigma,\alpha))$, and we call *sender-triggered out-of-equilibrium beliefs* the beliefs held by R which are activated by signals in $(Z,X)^{o2}((\zeta,\mu),(r,\sigma,\alpha))$. So, in a SCAS-IS game a *refinement acting on out-of-equilibrium beliefs* is a refinement that rules away equilibria by restricting admissible beliefs to a subset of those possibly activated by signals in $(Z,X)^{o}((\zeta,\mu),(r,\sigma,\alpha))$. Note also that – as it happens in SCAS games – although such refinements require that R observes an unexpected signal, they can potentially act on beliefs activated by all $(z,x) \in (Z,X)^{o}((\zeta,\mu),(r,\sigma,\alpha))$, i.e., on both sender-triggered and receiver-triggered out-of-equilibrium.

PROPOSITION **5.** *The SCAS-IS game* $\Gamma(T,p,\hat{U},v,c_x,c_z)$ *has a pooling equilibrium where all types pool on the same pair of signals. If* $((\zeta^P,\mu^P),(r^P,\sigma^P,\alpha^P))$ *is such a kind of pooling equilibrium, then it must be such that* $\zeta^P(t) = 0$ *and* $\mu^P(t) = 0$ *for all* $t \in T$, $r^P = r1$, $\sigma^P(z,r1) = s1$, *and* $\alpha^P(x,s1,r1,z) = \rho^{s1,r1}(\mu^P)$ *for all* $x \in X$ *and* $z \in Z$. *Moreover,* $((\zeta^P,\mu^P),(r^P,\sigma^P,\alpha^P))$ *survives any possible equilibrium refinement acting on out-of-equilibrium beliefs.*

Proof. We first show that $((\zeta^P,\mu^P),(r^P,\sigma^P,\alpha^P))$ is an equilibrium. Preliminarily, note that by B8 (uniqueness of best action under $s1$ and $r1$) R's expected utility $\sum_{t \in T} p_t v(t,\mu^P(t),y)$ admits a maximum over Y, denoted by y^*, and, hence, the profile $((\zeta^P,\mu^P),(r^P,\sigma^P,\alpha^P))$ exists.

Consider R deviating from (r^P,σ^P,α^P). Since $\alpha^P(x,s1,r1,z) = y^* = \rho^{s1,r1}(\mu)$ for all $x \in X$ and $z \in Z$, no strictly profitable deviation from α^P exists as long as R maintains $\sigma(z,r1) = s1$ for all $z \in Z$ and chooses $r1$. Consider a deviation to (r',σ',α') with $r' = r2$, and some $\sigma' \in \Sigma$ and $\alpha' \in \mathcal{A}^{IS}$. We observe that, since $\zeta^P(t) = 0$ and $\mu^P(t) = 0$ for all $t \in T$, R obtains no additional information by playing $r2$ instead of $r1$ and σ' instead of σ^P, and therefore her posterior beliefs must be equal to her priors p. So, by B6 and B7 (fixed positive acquisition cost of x and z, respectively), it follows that R's expected utility for playing (r',σ',α') is:

- if $\sigma'(0,r2) = s2$, $\mathbb{E}V[\sigma',\alpha'|r2,(\zeta^P,\mu^P)] = \sum_{t \in T} p_t v(t,0,\alpha'(0,s2,r2,0)) - c_x - c_z$;

- if $\sigma'(0,r2) = s1$, $\mathbb{E}V[\sigma',\alpha'|r2,(\zeta^P,\mu^P)] = \sum_{t \in T} p_t v(t,0,\alpha'(0,s1,r2,0)) - c_z$.

By B8 (uniqueness of best action under $s1$ and $r1$) we have that $\sum_{t \in T} p_t v(t,0,\alpha'(0,s2,r2,0))$ and $\sum_{t \in T} p_t v(t,0,\alpha'(0,s1,r2,0))$ are both not greater than $\sum_{t \in T} p_t v(t,0,y^*)$, implying that y^* is an optimal action when $r' = r2$ and σ' are played against (ζ^P,μ^P). So, by B6 and B7 (fixed positive acquisition cost of x and z, respectively), it follows that R's expected utility is lower under deviation $(r2,\sigma',\alpha')$ than under (r^P,σ^P,α^P), for all $\sigma' \in \Sigma$ and all $\alpha' \in \mathcal{A}^{IS}$.

Consider S deviating from (ζ^P,μ^P). In particular, consider S deviating to (ζ',μ') such that either $\zeta'(t') > 0$ for some $t' \in T$ or $\mu'(t') > 0$ for some $t' \in T$, or both. Recall that $\alpha^P(x,s1,r1,z) = y^*$ for all $x \in X$ and all $z \in Z$, i.e., the action chosen by R is y^* independently of the actual value

of $\zeta'(t)$ and $\mu'(t)$, $t \in T$. This, together with assumption B3 (costly signaling) implies that S's expected utility cannot be greater under any $(\zeta', \mu') \in \mathcal{Z} \times \mathcal{M}$ than under (ζ^P, μ^P).

We now show that no pooling equilibrium other than $((\zeta^P, \mu^P), (r^P, \sigma^P, \alpha^P))$ exists. Consider the profile $((\zeta^{P'}, \mu^{P'}), (r^{P'}, \sigma^{P'}, \alpha^{P'}))$ where $\zeta^{P'}(t) = z^{P'} \geq 0$ and $\mu^{P'}(t) = x^{P'} \geq 0$ for all $t \in T$, with $z^{P'}$ and $x^{P'}$ not both zero. Note that, exactly because $\zeta^{P'}(t) = z^{P'}$ and $\mu^{P'}(t) = x^{P'}$ for all $t \in T$, along the equilibrium path R never learns anything and so R takes the same action $y^{P'} = \alpha^{P'}(\mu^{P'}(t), s^{P'}, r^{P'}, \zeta^{P'}(t))$ for all $t \in T$. By assumptions B6 and B7 (fixed positive acquisition cost of x and z, respectively), R's expected utility is:

- if $r^{P'} = r2$ and $\sigma^{P'}(z, r2) = s2$ for all $z \in Z$, $\mathbb{E}V[\sigma^{P'}, \alpha^{P'}|r2, (\zeta^{P'}, \mu^{P'})] = \sum_{t \in T} p_t v(t, x^{P'}, y^{P'}) - c_x - c_z$;

- if $r^{P'} = r2$ and $\sigma^{P'}(z, r2) = s1$ for all $z \in Z$, $\mathbb{E}V[\sigma^{P'}, \alpha^{P'}|r2, (\zeta^{P'}, \mu^{P'})] = \sum_{t \in T} p_t v(t, x^{P'}, y^{P'}) - c_z$;

- if $r^{P'} = r1$ and $\sigma^{P'}(z, r1) = s2$ for all $z \in Z$, $\mathbb{E}V[s2, \alpha^{P'}|r1, (\zeta^{P'}, \mu^{P'})] = \sum_{t \in T} p_t v(t, x^{P'}, y^{P'}) - c_x$;

- if $r^{P'} = r1$ and $\sigma^{P'}(z, r1) = s1$ for all $z \in Z$, $\mathbb{E}V[s1, \alpha^{P'}|r1, (\zeta^{P'}, \mu^{P'})] = \sum_{t \in T} p_t v(t, x^{P'}, y^{P'})$.

These expected utilities imply that R strictly prefers to play, instead of $(r^{P'}, \sigma^{P'}, \alpha^{P'})$, any strategy $(r^{P''}, \sigma^{P''}, \alpha^{P''})$ such that $r^{P''} = r1$, $\sigma^{P''}(z) = s1$ for all $z \in Z$, and $\alpha^{P''} \in \mathcal{A}^{IS}$ such that $\alpha^{P''}(x, s1, r1, z) = y^{P'}$ for all $x \in X$ and $z \in Z$. So, in order for $(r^{P'}, \sigma^{P'}, \alpha^{P'})$ to be a best reply for R to $(\zeta^{P'}, \mu^{P'})$, it must be that $r^{P'} = r1$ and $\sigma(z)^{P'} = s1$ for all $z \in Z$; hence, $\alpha^{P'}(x, s1, r1, z)$ must be constant over X and Z, and in particular it must be such that $\alpha^{P'}(x, s1, r1, z) = \rho^{s1, r1}(\mu^{P'})$ for all $x \in X$ and $z \in Z$. But if this is the case, then S must have a profitable deviation. In particular, consider S deviating to (ζ^P, μ^P). Since R always responds with $\rho^{s1, r1}(\mu^{P'})$, it follows by B3 (costly signaling) that S's expected utility is strictly greater under (ζ^P, μ^P) than under $(\zeta^{P'}, \mu^{P'})$.

Finally, we show that $((\zeta^P, \mu^P), (r^P, \sigma^P, \alpha^P))$ survives any possible equilibrium refinement acting on out-of-equilibrium beliefs. Since R plays $r^P = r1$ and $\sigma(z)^P = s1$ for all $z \in Z$, it follows that $(Z, X)^e((\zeta^P, \mu^P), (r^P, \sigma^P, \alpha^P)) = (Z, X)^{o2}((\zeta^P, \mu^P), (r^P, \sigma^P, \alpha^P)) = \emptyset$ and $(Z, X)^{o1}((\zeta^P, \mu^P), (r^P, \sigma^P, \alpha^P)) = \mathbb{S}$, because R can observe a pair in $\mathbb{S}$ only if she deviates from her equilibrium strategy. In particular, no sender-triggered out-of-equilibrium belief exists because, since $\sigma(z)^P = s_1$ for all $z \in Z$, any pair in $\mathbb{S}$ that is chosen by S leads to the same R's information set, which is on the equilibrium path. Hence, at this information set, R must have constant beliefs which are identical to the priors p and which cannot be refined away by refinements acting on out-of-equilibrium beliefs.

So, refinements acting on out-of-equilibrium beliefs can rule out only beliefs associated with information sets that become active when a pair $(Z, X)^{o1}((\zeta^P, \mu^P), (r^P, \sigma^P, \alpha^P)) = \mathbb{S}$ is observed.

However, none of these receiver-triggered out-of-equilibrium beliefs is necessary to sustain the considered equilibrium. To see why, note that R always uses the priors p and S's strategy (ζ^P, μ^P) to evaluate whether to deviate or not from $(r^P, \sigma^P \alpha^P)$, as $(r^P, \sigma^P \alpha^P)$ is a best response to (ζ^P, μ^P) given p, no matter what are the receiver-triggered out-of-equilibrium beliefs held by R; since (ζ^P, μ^P) is also a best response to $(r^P, \sigma^P \alpha^P)$, it follows that S has no strictly profitable deviation from (ζ^P, μ^P), and this is again independent of the receiver-triggered out-of-equilibrium beliefs held by R. Hence, no refinement acting on out-of-equilibrium beliefs can refine away the pooling equilibrium $((\zeta^P, \mu^P), (r^P, \sigma^P, \alpha^P))$. $\qquad\square$

B The SCAS game with Acquisition Effort (SCAS-AE)

Consider a SCAS game where R, instead of choosing $s \in \{s1, s2\}$, can choose an acquisition effort $e \in [0, 1]$. The signal x is acquired by R with probability e.

In a SCAS-AE game, utility for S is $U : T \times X \times Y \to \mathbb{R}$, as a in SCAS game. Instead, utility for R is $V : T \times X \times Y \times [0, 1] \to \mathbb{R}$. The following assumptions on utility functions hold:

C1. *continuity:* U and V are continuous over x, y and e;

C2. *monotonicity in action:* U is strictly increasing in y;

C3. *costly signaling:* U is strictly decreasing in x;

C4. *single-crossing property:* $U(t, x, y) \leq U(t, x', y')$, with $x' > x$, implies that $U(t', x, y) < U(t', x', y')$ for all $t' > t$ and $y, y' \in Y$;

C5. *costly effort of acquisition:* V is strictly decreasing in e;

C6. *separability of the effort cost:* $V(t, x, y, 0) - V(t, x, y, e) = c_e(e)$.

In the light of C5 and C6, we have that $V(t, x, y, e) = v(t, x, y) - c_e(e)$.

In a SCAS-AE game, a strategy for S is a function $\mu \in \mathcal{M}$ describing, as in a SCAS game, a type's choice of x. A strategy for R is a triple $(e, \alpha 1, \alpha 2)$ where:

- $e \in [0, 1]$ describes R's choice of acquisition effort;

- $\alpha 1 \in Y$ describes R's choice of action if the signal x is not acquired, and hence is unconditional on x;

- $\alpha 2 : X \to Y$ describes R's choice of action if the signal x is acquired, and hence is conditional on x; $\mathcal{A}^{AE}$ denotes the set of possible functions $\alpha 2$.

For given μ and $(e, \alpha 1, \alpha 2)$, R has posterior beliefs that depend on whether the signal x has been acquired or not. Let us indicate, with a slight abuse of notation, the event "x is not acquired" with $s = s1$ and the event "x is acquired" with $s = s2$. If $s = s2$ then R has posterior beliefs $\beta(x|\mu, s2) = (\beta_1(x|\mu, s2), \ldots, \beta_n(x|\mu, s2)) \in \Delta T$, where $\beta_t(x|\mu, s2)$ denotes the probability that S is of type t, conditional on the observation of x. These beliefs can be obtained by Bayes's rule, if applicable, or be chosen otherwise. If, instead, $s = s1$ then R can only rely on her priors – no new information is acquired – so that posteriors are identical to priors: $\beta_t(x|\mu, s1) = \beta_t(x'|\mu, s1) = p_t$, for all $t \in T$ and all $x, x' \in X$.

We also introduce the following assumption:

C7. *uniqueness of best action when signal x is not acquired:*
   $\rho^{s1}(\mu) = \arg \max_{y \in Y} \sum_{t \in T} p_t v(t, \mu(t), y)$ *exists and is single valued.*

Assumption C7 is the counterpart of assumption A6 in a SCAS game. Moreover, because of assumption C6, assumption C7 implies that $\rho^{s1}(\mu) = \arg \max_{y \in Y} \sum_{t \in T} p_t v(t, \mu(t), y)$, i.e., $\rho^{s1}(\mu)$ is the best reply whenever posteriors are identical to priors and independently of the choice of e.

DEFINITION **3.** *(weak Perfect Bayes-Nash equilibrium of the SCAS-AE game)*
A wPBE equilibrium of a SCAS-AE game is a profile of strategies $(\mu, (e, \alpha 1, \alpha 2))$ such that:

G1. $\mu(t) \in \underset{x \in X}{\arg \max} \ [(1 - e)U(t, x, \alpha 1) + eU(t, x, \alpha 2(x))]$, *for all $t \in T$;*

G2. for all $x \in X$, there exists beliefs $\beta(x|\mu, s) \in \Delta T$ such that $(\mu, (e, \alpha 1, \alpha 2))$ satisfies:

   G2.1. $\alpha 1 = \rho^{s1}(\mu)$;

   G2.2. $\alpha 2(x) \in \underset{y \in Y}{\arg \max} \sum_{t \in T} \beta_t(x|\mu, s2) v(t, x, y)$ *for all $x \in X$;*

   G2.3. $e \in \underset{e \in [0,1]}{\arg \max} \sum_{t \in T} p_t \left[(1 - e)V(\mu(t), \alpha 1, e) + e \left(\sum_{k \in T} \beta_k(\mu(t)|\mu, s) V(t, \mu(t), \alpha 2(\mu(t)), e) \right) \right]$;

G3. the beliefs $\beta(x|\mu, s) \in \Delta T$ are calculated by means of Bayes's rule whenever possible.

The meaning of G1 is straightforward: S must be best-replying to R, taking into account that R acquires signal x with probability e. Similarly, the meaning of G2 is that R must be best-replying to S given her beliefs, taking into account that signal x is acquired with probability e. Condition G3 is also straightforward. We observe that, in the present setup, posterior beliefs along the equilibrium path are the following:

- if $s = s2$ then, for all $t \in T$ and for all x such that $\mu(t') = x$ for some $t' \in T$:

$$\beta_t(x|\mu, s) = \begin{cases} \dfrac{p_t}{\sum_{k:\mu(k)=x} p_k} & \text{if } \mu(t) = x \\ 0 & \text{if } \mu(t) \neq x; \end{cases}$$

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• if $s = s1$ then $\beta_t(x|\mu, s) = p_t$ for all $t \in T$ and for all $x \in X$.

For comparability purposes, we observe that for any given SCAS game $\Gamma(T, p, U, v, c)$ we have, besides the associated SFAS game $\Gamma(T, p, U, v, 0)$, also an associated SCAS-AE game, that we denote with $\Gamma(T, p, U, v, c_e)$.

We now turn our attention to the relevant beliefs in a SCAS-AE game. For given strategy profile $(\mu, (e, \alpha1, \alpha2))$ and priors p, R has beliefs $\beta(x|\mu, s) \in \Delta T$ associated with each of her information sets where an action in Y has to be chosen. Denote with $X^e(\mu, (s, \alpha1, \alpha2)) \subseteq X$ the set of signals that R can observe on information sets along the equilibrium path, i.e., at information sets that contain decision nodes along the equilibrium path. Denote with $X^o(\mu, (s, \alpha1, \alpha2)) = X \backslash X^e(\mu, (e, \alpha1, \alpha2))$ the set of signals that R can observe only at information sets off the equilibrium path, i.e., at information sets that do not contain decision nodes lying on the equilibrium path.

Moreover, denote with $X^{o1}(\mu, (e, \alpha1, \alpha2)) \subseteq X^o(\mu, (e, \alpha1, \alpha2))$ the set of signals off the equilibrium path that R cannot observe as a consequence of S deviating from μ because a deviation by R is required. Also, denote with $X^{o2}(\mu, (e, \alpha1, \alpha2)) = X^o(\mu, (e, \alpha1, \alpha2)) \backslash X^{o1}(\mu, (e, \alpha1, \alpha2))$ the set of signals off the equilibrium path that R can potentially observe as a consequence of S deviating from μ. Similarly for what done for CAS game, we call *receiver-triggered out-of-equilibrium beliefs* the beliefs held by R which are activated by signals in $X^{o1}(\mu, (e, \alpha1, \alpha2))$, and we call *sender-triggered out-of-equilibrium beliefs* the beliefs held by R which are activated by signals in $X^{o2}(\mu, (e, \alpha1, \alpha2))$. So, a refinement that rules away equilibria by restricting admissible beliefs to a subset of those possibly activated by signals in $X^o(\mu, (e, \alpha1, \alpha2))$ can be regarded as a *refinement acting on out-of-equilibrium beliefs.* Note also that – as it happens in SCAS games – although such refinements require that R observes an unexpected signal, they can potentially act on beliefs activated by all $x \in X^o(\mu, (e, \alpha1, \alpha2))$, i.e., they act not only on sender-triggered out-of-equilibrium beliefs, but also on receiver-triggered ones.

PROPOSITION **6.** *The SCAS-AE game* $\Gamma(T, p, U, v, c_e)$ *has a pooling equilibrium. If* $(\mu^P, (e^P, \alpha1^P, \alpha2^P))$ *is a pooling equilibrium, then it must be such that* $\mu^P(t) = 0$ *for all* $t \in T$, $e^P = 0$, $\alpha1^P = \alpha2^P(\mu^P(t)) = \rho^{s1}(\mu^P)$ *for all* $t \in T$. *Moreover,* $(\mu^P, (e^P, \alpha1^P, \alpha2^P))$ *survives any possible equilibrium refinement acting on out-of-equilibrium beliefs.*

Proof. We first show that the profile $(\mu^P, (e^P, \alpha1^P, \alpha2^P))$ is an equilibrium. Preliminarily, note that by C7 (uniqueness of best action when signal x is not acquired) R's expected utility, i.e., $\sum_{t \in T} p_t v(t, \mu^P(t), y) = \sum_{t \in T} p_t V(t, \mu^P(t), y, 0)$, admits a maximum over Y, denoted by y^*, and so the profile $(\mu^P, (e^P, \alpha1^P, \alpha2^P))$ exists.

Consider R deviating from $(e^P, \alpha1^P, \alpha2^P)$. Since $\alpha2^P(x) = y^* = \rho^{s1}(\mu)$ for all $x \in X$, no strictly profitable deviation from α^P exists as long as $s = s1$ takes place. But if R maintains $e^P = 0$ then

$s = s1$ takes place with probability 1. So, consider a deviation to $(e', \alpha1', \alpha2')$ with $e' > 0$ and some $\alpha1' \in Y$ and $\alpha2' \in \mathcal{A}^{AE}$. We observe that, since $\mu^P(t) = 0$ for all $t \in T$, R obtains no additional information if the event $s = s2$ takes places, and therefore her posterior beliefs in such a case must still be equal to her priors p. By C6 (separability of the effort cost) the expected utility for playing $(e^P, \alpha1^P, \alpha2^P)$ is $\sum_{t \in T} p_t v(t, 0, y^*)$ while the expected utility for playing $(e', \alpha1', \alpha2')$ is $\sum_{t \in T} p_t [e' v(t, 0, \alpha1') + (1 - e')v(t, 0, \alpha2'(0))] - c_e(e')$. By C7 (uniqueness of best action when signal x is not acquired) it follows that $v(t, 0, y^*) \geq v(t, 0, \alpha1')$ and $v(t, 0, y^*) \geq v(t, 0, \alpha2'(0))$ which implies that $v(t, 0, y^*) \geq [e' v(t, 0, \alpha1') + (1 - e')v(t, 0, \alpha2'(0))]$, and hence by C5 (costly effort of acquisition) we get that R's expected utility is not greater under deviation $(e', \alpha1', \alpha2')$ than under $(e^P, \alpha1^P, \alpha2^P)$, for all $\alpha1' \in Y$ and all $\alpha2 \in \mathcal{A}^{AE}$.

Consider S deviating from μ^P. In particular, consider S deviating to μ' such that $\mu'(t') > 0$ for some $t' \in T$. Recall that $e^P = 0$ and $\alpha1^P = \alpha2^P(x) = y^*$ for all $x \in X$, i.e., signal x is acquired with probability 0 and the action chosen by R is y^* independently of the actual value of $\mu'(t)$, $t \in T$. These facts, together with assumption C3 (costly signaling), imply that S's expected utility is equal, for each $t \in T$, to $U(t, \mu(t), y^*)$ and, therefore, it cannot be greater than $U(t, \mu^P(t), y^*)$ for any $\mu' \in \mathcal{M}$.

We now show that no pooling equilibrium other than $(\mu^P, (e^P, \alpha1^P, \alpha2^P))$ exists. Consider an alternative pooling profile $(\mu^{P'}, (e^{P'}, \alpha1^{P'}, \alpha2^{P'}))$ where $\mu^{P'}(t) = x^{P'} > 0$ for all $t \in T$. Note that, since $\mu^{P'}(t) = x^{P'}$ for all $t \in T$, along the equilibrium path R never learns anything. So, if event $s = s1$ takes place and x is acquired, R must take the same action $y^{P'} = \alpha2^{P'}(\mu^{P'}(t))$ for all $t \in T$. Hence, by assumption C6 (separability of the effort cost), R's gets an expected utility equal to $\sum_{t \in T} p_t \left[e^{P'} v(t, x^{P'}, y^{P'}) + (1 - e^{P'})v(t, x^{P'}, \alpha1^{P'}) \right] - c_e(e^{P'})$ which, by C5 (costly effort of acquisition) and non-negativity of $e^{P'}$, is also strictly lower than the maximum between $\sum_{t \in T} p_t v(t, x^{P'}, y^{P'})$ and $\sum_{t \in T} p_t v(t, x^{P'}, \alpha1^{P'})$, each of which is in turn not greater than the expected utility of playing $(e^{P''}, \alpha1^{P''}, \alpha2^{P''})$ such that $e^{P''} = 0$ and $\alpha1^{P''} = \alpha2^{P''}(x) = \rho^{s1}(\mu^{P'})$, because by C7 (uniqueness of best action when signal x is not acquired) $\rho^{s1}(\mu^{P'})$ maximizes $\sum_{t \in T} p_t v(t, x^{P'}, y)$ with respect to $y \in Y$. So, in order for $(e^{P'}, \alpha1^{P'}, \alpha2^{P'})$ to be a best reply for R to $\mu^{P'}$, it must be that $e^{P'} = 0$ and $\alpha1^{P'} = \alpha2^{P'}(x) = \rho^{s1}(\mu^{P'})$ for all $x \in X$. But if this is the case, then we claim that S has a profitable deviation. In particular, consider S deviating to μ^P. Since R always responds with $\rho^{s1}(\mu^{P'})$ and by B3 (costly signaling), it follows that S's expected utility is strictly greater under μ^P than under $\mu^{P'}$.

Finally, we show that $(\mu^P, (e^P, \alpha1^P, \alpha2^P))$ survives any possible equilibrium refinement acting on out-of-equilibrium beliefs. Since R plays $e^P = 0$, the event $s = s1$ takes place with probability 1 along the equilibrium path. So, it follows that $X^e(\mu^P, (e^P, \alpha1^P, \alpha2^P)) = X^{o2}(\mu^P, (e^P, \alpha1^P, \alpha2^P)) = \emptyset$ and $X^{o1}(\mu^P, (e^P, \alpha1^P, \alpha2^P)) = X$, because R can observe a signal $x \in X$ only if she deviates

from her equilibrium strategy. In particular, no sender-triggered out-of-equilibrium belief exists because, since $s = s_1$ with probability 1, any $x \in X$ that is chosen by S leads to the same R's information set, which is on the equilibrium path. Hence, at this information set, R must have constant beliefs which are identical to the priors p and which cannot be refined away by refinements acting on out-of-equilibrium beliefs.

Therefore, refinements acting on out-of-equilibrium beliefs can rule out only beliefs associated with information sets that become active when a signal in $x \in X^{o1}(\mu^P, (e^P, \alpha 1^P, \alpha 2^P)) = X$ is observed. However, none of these receiver-triggered out-of-equilibrium beliefs is necessary to sustain the considered equilibrium. To see why, note that R always uses the priors p and S's strategy μ^P to evaluate whether to deviate or not from $(e^P, \alpha 1^P, \alpha 2^P)$; so, there is no deviation by S that can induce R to deviate from $(e^P, \alpha 1^P, \alpha 2^P)$, as $(e^P, \alpha 1^P, \alpha 2^P)$ is a best response to μ^P given p, no matter what are the receiver-triggered out-of-equilibrium beliefs held by R; since also μ^P is a best response to $(e^P, \alpha 1^P, \alpha 2^P)$, it follows that S has no strictly profitable deviation from μ^P, and this is again independent of the receiver-triggered out-of-equilibrium beliefs held by R. Hence, no refinement acting on out-of-equilibrium beliefs can refine away the considered equilibrium $(\mu^P, (e^P, \alpha 1^P, \alpha 2^P))$. $\qquad\square$